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Polynomial Interpolation and Hyperinterpolation over General Regions

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This paper studies a generalization of polynomial interpolation: given a continuous function over a rather general manifold, hyperinterpolation is a linear approximation that makes use of values of f on a well chosen finite set. The approximation is a discrete least-squares approximation constructed with the aid of a high-order quadrature rule: the role of the quadrature rule is to approximate the Fourier coefficients of f with respect to an orthonormal basis of the space of polynomials of degree $\leqslant n$. The principal result is a generalization of the result of Erdös and Turàn for classical interpolation at the zeros of orthogonal polynomials: for a rule of suitably high order (namely $2n$ or greater), the L_2 error of the approximation is shown to be within a constant factor of the error of best uniform approximation by polynomials of degree $\leqslant n$. The L_2 error therefore converges to zero as the degree of the approximating polynomial approaches ∞. An example discussed in detail is the approximation of continuous functions on the sphere in $\mathbb{R}^s$ by spherical polynomials. In this case the number of quadrature points must exceed the number of degrees of freedom if $n > 2$ and $s \geqslant 3$. In such a situation the classical interpolation property cannot hold, whereas satisfactory hyperinterpolation approximations do exist. © 1995 Academic Press, Inc.

1. INTRODUCTION

A theorem of Erdös and Turàn [8] states: if r is a classical weight function on $[-1, 1]$ (i.e. r non-negative, integrable, and vanishing only on a finite set), and $\{p_n\}$ is a family of orthogonal polynomials with respect to r, with p_n of degree n, the polynomial $L_n f$ of degree n or less that interpolates a continuous function f at the zeros of p_{n+1} satisfies

$$\left(\int_{-1}^{1} [L_n f(x) - f(x)]^2\, r(x)\, dx\right)^{1/2} \leqslant 2\left(\int_{-1}^{1} r(x)\, dx\right)^{1/2} E_n(f),$$

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where

$$E_n(f) = \inf_{\chi \in \mathbb{P}_n} \|f - \chi\|_\infty .$$

A 2-dimensional generalization has been discussed recently by Xu [28, Section 4].

In this paper we study a more general class of linear polynomial approximations. The results encompass the classical case of interpolation at the zeros of orthogonal polynomials on an interval, but also extend to more general geometries, such as the sphere in any number of dimensions, and to situations in which the number of data points exceeds the number of degrees of freedom of the approximating polynomial. Under suitable hypotheses we are able to prove a theorem analogous to that of Erdös and Turàn [8]: that *the L_2 error, in an appropriate sense, is bounded, to within a constant factor, by the error of best uniform approximation.* The L_2 error therefore converges to zero as the degree of the approximating polynomial approaches ∞; and the convergence is rapid if f is suitably smooth.

The typical situation in which this approximation can be useful is that in which f is a smooth, analytically specified function whose values are computable at any desired point. The approximation generally has no role in situations in which f is an experimentally measured quantity, or one whose values are determined by mechanical or freehand design.

The approximation is constructed with the aid of a well chosen quadrature rule of suitably high order: the quadrature rule is used to approximate the Fourier coefficients with respect to an arbitrary orthonormal set. If the number of quadrature points equals the number of degrees of freedom, i.e. if the quadrature rule is "minimal," then the approximating polynomial interpolates f at the quadrature points. But if the number of quadrature points exceeds the number of degrees of freedom then the classical interpolation conditions are generally not satisfied. In that case we do not speak of interpolation, but rather of "hyperinterpolation," intending to suggest interpolation attempted with too many points.

In the particular case of a circle, and hence of approximation by trigonometric polynomials, an appropriate quadrature rule is the (periodic) rectangle rule on a uniform mesh. In this case Zygmund [29, Chapter 10, Theorem 7] has already established the L_2 convergence of the hyperinterpolation approximation (see Section 3). The present work may be regarded as an extension of Zygmund's result to more general surfaces and regions. A different extension, to the approximation of multiply-periodic functions by s-dimensional trigonometric polynomials, is discussed by Hua and Wang [10, Chapter 9].

We shall pay particular attention to the problem of approximation by spherical polynomials on the unit sphere in $\mathbb{R}^s$. One can imagine a variety of reasons for wanting to approximate a smooth function f on the sphere by a polynomial. For example, one might want to integrate f over part of the sphere, or carry out other numerical procedures more easily performed on the polynomial. Our approximation is in this case generally not interpolatory, because, as we shall see, in most cases the number of quadrature points must exceed the number of degrees of freedom of the approximating space.

Classical interpolation by polynomials over the sphere has been considered by Reimer [19]. In particular, Reimer estimates the Lebesgue constants via the sums of the squares of the Lagrangian interpolation polynomials. The difficulty lies in finding an appropriate set of interpolation points. For example, for the unit sphere in $\mathbb{R}^3$ there are $(n+1)^2$ linearly independent spherical polynomials of degree $\geqslant n$, yet few of the point sets on the sphere which are persuasively well distributed have a number of points which is exactly a perfect square.

In the next section the theory is developed in a general setting, with the main result being stated as Theorem 1. Then in Section 3 the theory is applied to three examples: the interval, the circle, and the sphere in $\mathbb{R}^s$. The final section discusses in more detail the case of the sphere.

2. The General Theory

Let Ω be a bounded region of $\mathbb{R}^s$ which is either the closure of a connected open domain, or a smooth closed lower-dimensional manifold in $\mathbb{R}^s$. The region Ω is assumed to have finite measure with respect to a given (positive) measure $d\omega$, that is

$$\int_\Omega d\omega = V < \infty. \tag{2.1}$$

We wish to approximate $f \in C(\Omega)$ by a polynomial on Ω (or, in the case of a lower dimensional manifold, by the restriction of a polynomial to the manifold; even in this case we shall continue to say just "polynomial"). The approximating polynomial $L_n f$ is to be linear in f, and to be of degree $\leqslant n$. Thus $L_n f$ is to belong to S_n, where S_n is the space of polynomials on Ω of degree $\leqslant n$.

To motivate the coming definition of $L_n f$, it is useful to define first $P_n f$, the orthogonal projection of f onto S_n with respect to the inner product

$$(v, z) = \int_\Omega vz \, d\omega. \tag{2.2}$$

Let $d = d_n = \dim S_n$, and let

$$\{p_1, \ldots, p_d\} \subset S_n$$

be an orthonormal basis of S_n, that is

$$(p_i, p_j) = \delta_{ij}, \qquad 1 \leqslant i, j \leqslant d. \tag{2.3}$$

Then

$$P_n f = \sum_{j=1}^{d} (f, p_j) p_j. \tag{2.4}$$

As part of the definition of $L_n f$, it is assumed that we are given a quadrature rule of the form

$$\sum_{k=1}^{m} w_k g(x_k) \approx \int_\Omega g \, d\omega, \tag{2.5}$$

where

$$x_k \in \Omega \quad \text{and} \quad w_k > 0 \quad \text{for} \quad 1 \leqslant k \leqslant m, \tag{2.6}$$

with the property that the rule is exact for every polynomial of degree $\leqslant 2n$. Thus we require

$$\sum_{k=1}^{m} w_k g(x_k) = \int_\Omega g \, d\omega \qquad \forall g \in S_{2n}. \tag{2.7}$$

Corresponding to the inner product (2.2), it is useful to define a "discrete inner product"

$$(v, z)_m := \sum_{k=1}^{m} w_k v(x_k) z(x_k), \tag{2.8}$$

in which the exact integral is replaced by the quadrature rule. Then $L_n f$ is defined, analogously to $P_n f$, by:

DEFINITION.

$$L_n f := \sum_{j=1}^{d} (f, p_j)_m p_j. \tag{2.9}$$

This definition, like that of $P_n f$, is easily seen to be invariant under a change of the orthonormal basis $\{p_1, \ldots, p_d\}$.

Remark. While the assumed orthonormality of the basis is convenient theoretically, it is not necessary for practical applications. If $\{h_1, ..., h_d\}$ is a non-orthonormal basis then (2.9) is replaced by

$$L_n f = \sum_{i,j=1}^{d} (f, h_j)_m (H^{-1})_{ji} h_i,$$

where H is the $d \times d$ matrix with elements $(h_i, h_j)_m$.

A key role is played by a discrete orthogonality relation that mimics (2.3):

LEMMA 1. *For* $1 \leqslant i, j \leqslant d$,

$$(p_i, p_j)_m = \delta_{ij}. \tag{2.10}$$

Proof. Because $p_i p_j$ is a polynomial of degree $\leqslant 2n$, it follows from (2.7) and (2.3) that

$$(p_i, p_j)_m = (p_i, p_j) = \delta_{ij}. \quad \blacksquare \tag{2.11}$$

From this follows a known lower bound on the number of quadrature points (see, for example, [15]):

LEMMA 2. *If a quadrature rule is exact for all polynomials of degree* $\leqslant 2n$, *the number of quadrature points* m *satisfies* $m \geqslant d_n$.

Proof. Let Q be the $d \times m$ matrix with elements $q_{jk} = w_k^{1/2} p_j(x_k)$. Then (2.11) asserts that the rows of Q are orthogonal, and hence linearly independent, so that rank $Q = d$. Since rank $Q \leqslant m$, the result follows. $\blacksquare$

DEFINITION. An m-point quadrature rule that is exact for all polynomials of degree $\leqslant 2n$ is *minimal* if $m = d_n$.

Remark. Other lower bounds on the number of quadrature points are known for special situations, such as "central symmetry" of the rule, the region Ω and the measure. For a summary of such results see Mysovskikh [17].

The next result states that the approximation $L_n f$ has the classical interpolation property if and only if the quadrature rule is minimal:

LEMMA 3. *The classical interpolation formula*

$$L_n f(x_k) = f(x_k), \qquad 1 \leqslant k \leqslant m, \tag{2.12}$$

holds for arbitrary $f \in C(\Omega)$ *if and only if the quadrature rule is minimal.*

Proof. If $m = d$ then the matrix Q defined in the proof of Lemma 2 is square, and by Lemma 1 satisfies $QQ^T = I$. Together these imply $Q^TQ = I$, or

$$\sum_{j=1}^{d} p_j(x_k)\, p_j(x_l) = w_k^{-1}\delta_{kl}, \qquad 1 \leqslant k, l \leqslant m. \tag{2.13}$$

The desired property (2.12) now follows immediately from the definition (2.9).

In the reverse direction, (2.12) holding for arbitrary $f \in C(\Omega)$ implies that (2.13) holds, or $Q^TQ = I$. Since $QQ^T = I$ by Lemma 1, Q must be square, implying $m = d$. ∎

Remark. If the quadrature rule is minimal then (2.13) allows the weights to be expressed explicitly: setting $l = k$ it gives

$$w_k = \left(\sum_{j=1}^{d} p_j(x_k)^2 \right)^{-1}, \qquad 1 \leqslant k \leqslant d. \tag{2.14}$$

This formula for the weights in a "cubature formula with fewest nodes" has been obtained previously by Mysovskikh [15], using an argument equivalent to that in Lemma 3.

The next result asserts that $L_n f$ becomes exact if f is a polynomial of degree $\leqslant n$.

LEMMA 4. *If $f \in S_n$ then $L_n f = f$.*

Proof. Since $f \in S_n$ it may be expressed as

$$f = \sum_{j=1}^{d} a_j p_j,$$

giving, from (2.9),

$$\begin{aligned} L_n f &= \sum_{j=1}^{d} \left(\sum_{i=1}^{d} a_i p_i, p_j \right)_m p_j \\ &= \sum_{j=1}^{d} \sum_{i=1}^{d} a_i (p_i, p_j)_m p_j \\ &= \sum_{j=1}^{d} a_j p_j = f. \quad ∎ \end{aligned}$$

We are now ready to state the main result of the paper. The norms that appear in the theorem are defined by

$$\|g\|_2 = \left(\int g^2 \, d\omega \right)^{1/2},$$

$$\|g\|_\infty = \sup_{x \in \Omega} |g(x)|, \qquad g \in C(\Omega),$$

and $E_n(g)$ is the error of best uniform approximation of g by an element of S_n,

$$E_n(g) = \inf_{\chi \in S_n} \|g - \chi\|_\infty, \qquad g \in C(\Omega). \tag{2.15}$$

THEOREM 1. *Given $f \in C(\Omega)$, let $L_n f \in S_n$ be defined by* (2.9), *where the quadrature points and weights in the discrete inner product satisfy* (2.6) *and* (2.7). *Then*

$$\|L_n f\|_2 \leqslant V^{1/2} \|f\|_\infty, \tag{2.16}$$

and

$$\|L_n f - f\|_2 \leqslant 2V^{1/2} E_n(f). \tag{2.17}$$

Thus

$$\|L_n f - f\|_2 \to 0 \qquad as \quad n \to \infty.$$

Before proving the theorem it is convenient to prove first one more lemma.

LEMMA 5. *Under the conditions of Theorem* 1,

(a) $(f - L_n f, \chi)_m = 0 \ \forall \chi \in S_n$,

(b) $(L_n f, L_n f)_m + (f - L_n f, f - L_n f)_m = (f, f)_m$,

(c) $(L_n f, L_n f)_m \leqslant (f, f)_m$,

(d) $(f - L_n f, f - L_n f)_m = \min_{\chi \in S_n} (f - \chi, f - \chi)_m$.

Proof. (a) This follows immediately from the definition (2.9) and the discrete orthogonality relation (2.10).

(b) This follows from $(L_n f, L_n f)_m = (f, L_n f)_m$, which is a consequence of (a).

(c) This is immediate from (b), since $(g, g)_m \geqslant 0$.

(d) Replacing f by $f-\chi$ in (b), with $\chi \in S_n$, we have

$$(f-\chi, f-\chi)_m = (f - L_n f, f - L_n f)_m + (L_n f - \chi, L_n f - \chi)_m,$$

from which the result follows immediately. ∎

Remarks. Part (a) of the lemma asserts that the m-vector $\{L_n f(x_k)\}_{k=1}^m$ is the orthogonal projection, with respect to the inner product $(\cdot, \cdot)_m$, of the m-vector $\{f(x_k)\}_{k=1}^m$, onto the d-dimensional space spanned by $\{p_j(x_k)\}_{k=1}^m$, $1 \leq j \leq d$. Part (b) is the corresponding Pythagoras theorem. Part (c), a trivial corollary, is the crucial result for proving Theorem 1. Part (d) gives us an interpretation of $L_n f$, as the best discrete least-squares approximation (weighted by the quadrature weights) of f at the quadrature points.

Proof of Theorem 1. The stability result (2.16) follows from

$$\begin{aligned}\|L_n f\|_2^2 &= (L_n f, L_n f) = (L_n f, L_n f)_m \leq (f, f)_m \\ &= \sum_{k=1}^m w_k f(x_k)^2 \leq \sum_{k=1}^m w_k \|f\|_\infty^2 = V \|f\|_\infty^2,\end{aligned}$$

where in the first step we used (2.7), which is applicable because $(L_n f)^2 \in S_{2n}$, then Lemma 5(c), and in the last step again (2.7), this time with $g = 1$.

The error bound (2.17) then follows by a standard argument: for any $\chi \in S_n$ we have, with the aid of Lemma 4 and the first part of the theorem,

$$\begin{aligned}\|L_n f - f\|_2 &= \|L_n(f-\chi) - (f-\chi)\|_2 \leq \|L_n(f-\chi)\|_2 + \|f-\chi\|_2 \\ &\leq V^{1/2} \|f-\chi\|_\infty + V^{1/2} \|f-\chi\|_\infty = 2V^{1/2} \|f-\chi\|_\infty.\end{aligned}$$

It follows, since this holds for arbitrary $\chi \in S_n$, that

$$\|L_n f - f\|_2 \leq 2V^{1/2} \inf_{\chi \in S_n} \|f-\chi\|_\infty = 2V^{1/2} E_n(f). \quad ∎$$

Remarks. If $L_n f$ has the classical interpolation property (2.12) then the proof of Theorem 1 simplifies, in that the first inequality is replaced by the trivial equality $(L_n f, L_n f)_m = (f, f)_m$.

The error estimate in the theorem may be used to derive error estimates for other quantities. One interesting application is to a generalization of the method of "product integration;" see [21, 22] for the classical 1-dimensional case. In this method an integral over Ω of the form $\int_\Omega hf\,d\omega$, where

f is smooth and h contains any singularities in the integrand, is approximated by

$$\int_\Omega hL_n f\,d\omega = \sum_{j=1}^{d} (f, p_j)_m \int_\Omega hp_j\,d\omega = \sum_{k=1}^{m} W_k f(x_k),$$

where

$$W_k = w_k \sum_{j=1}^{d} p_j(x_k) \int_\Omega hp_j\,d\omega, \qquad k = 1, ..., m. \tag{2.18}$$

The following corollary of Theorem 1 gives a useful error bound for this approximation:

COROLLARY. *Under the conditions in Theorem* 1, *let h be measurable on Ω with respect to $d\omega$ and satisfy $\|h\|_2 < \infty$, and let $W_1, ..., W_m$ be given by* (2.18). *Then*

$$\left| \sum_{k=1}^{m} W_k f(x_k) - \int_\Omega hf\,d\omega \right| \leqslant 2\,\|h\|_2\, V^{1/2} E_n(f). \tag{2.19}$$

Proof. From the Cauchy-Schwarz inequality and Theorem 1,

$$\left| \sum_{k=1}^{m} W_k f(x_k) - \int_\Omega hf\,d\omega \right| = \left| \int_\Omega h(L_n f - f)\,d\omega \right|$$
$$\leqslant \|h\|_2\, \|L_n f - f\|_2 \leqslant \|h\|_2\, 2V^{1/2} E_n(f). \qquad \blacksquare$$

For example, we may take h to be the characteristic function of a subset Ω_1 of Ω, so that $\int hf\,d\omega$ is the integral of f over Ω_1. A curiosity in this case is that the approximation $\sum_k W_k f(x_k)$ involves values of f at points at which the whole integrand hf is zero.

3. EXAMPLES

We begin with simple 1-dimensional examples.

EXAMPLE A (The interval). Here

$\Omega = [-1, 1]$,

$d\omega = r(x)\,dx$, with $r \in L(-1, 1)$, $r(x) \geqslant 0$, $r(x) = 0$ only on a finite set,

$S_n = \mathbb{P}_n[-1, 1]$,

$d = n + 1$.

Let $\{p_j\colon j \geqslant 0\}$ be a system of normalized orthogonal polynomials on $[-1, 1]$ with respect to the weight r, i.e. p_j is of degree j, and

$$\int_{-1}^{1} p_i(x)\, p_j(x)\, r(x)\, dx = \delta_{ij}.$$

For fixed $m \geqslant n+1$, let $\{x_k\}_{k=1}^{m}$ and $\{w_k\}_{k=1}^{m}$ be "Gauss" points and weights with respect to this system, i.e. x_k is the kth zero of p_m, and the weights are such that

$$\sum_{k=1}^{m} w_k\, g(x_k) = \int_{-1}^{1} g(x)\, r(x)\, dx \qquad \forall g \in \mathbb{P}_{2m-1}.$$

In this case our approximation (2.9) becomes

$$L_n f = \sum_{j=0}^{n} \sum_{k=1}^{m} w_k f(x_k)\, p_j(x_k)\, p_j. \tag{3.1}$$

By Theorem 1 the error in $L_n f$ satisfies

$$\left(\int_{-1}^{1} |L_n f(x) - f(x)|^2\, r(x)\, dx\right)^{1/2} \leqslant 2\left(\int_{-1}^{1} r(x)\, dx\right)^{1/2} \inf_{\chi \in \mathbb{P}_n} \|f - \chi\|_\infty. \tag{3.2}$$

For $m = n+1$ the approximation is interpolatory, by Lemma 3. Moreover, (2.14) with $m = d = n+1$ is a known expression for the Gauss weights (see [26, Theorem 3.4.2]). The result (3.2) in this case was obtained by Erdös and Turàn [8]. For $m > n+1$ the result (3.2) seems to be new. It tells us in effect that the Erdös-Turàn bound is not affected if we use in (3.1) a Gauss quadrature rule of higher order than necessary.

EXAMPLE B (The circle). Here

$\Omega =$ unit circle $\subset \mathbb{R}^2$,

$d\omega =$ angular measure in radians,

$V = \int d\omega = 2\pi$,

$S_n = \operatorname{span}\{1, \cos\theta, \sin\theta, \ldots, \cos n\theta, \sin n\theta\}$,

$d = 2n+1$.

For the quadrature rule we take the rectangle rule with m equal intervals of length $2\pi/m$. This rule is easily seen to be exact for all trigonometric polynomials of degree $\leqslant m-1$, i.e.

$$\frac{2\pi}{m}\sum_{k=0}^{m-1} g\left(\frac{2\pi k}{m}\right)=\int_0^{2\pi} g(\theta)\,d\theta \qquad \forall g\in S_{m-1},$$

so that (2.7) holds if $m \geqslant 2n+1$. The resulting approximation is

$$L_n f(\theta)=\tfrac{1}{2}a_0+\sum_{j=1}^{n}(a_j\cos j\theta+b_j\sin j\theta), \tag{3.3}$$

where

$$a_j=\frac{2}{m}\sum_{k=0}^{m-1}\cos\left(\frac{2\pi jk}{m}\right)f\left(\frac{2\pi k}{m}\right), \qquad j\geqslant 0, \tag{3.4}$$

$$b_j=\frac{2}{m}\sum_{k=1}^{m-1}\sin\left(\frac{2\pi jk}{m}\right)f\left(\frac{2\pi k}{m}\right), \qquad j\geqslant 1. \tag{3.5}$$

By Theorem 1 the approximation $L_n f$ satisfies, for $m\geqslant 2n+1$,

$$\left(\int_0^{2\pi}|L_n f(\theta)-f(\theta)|^2\,d\theta\right)^{1/2}\leqslant 2(2\pi)^{1/2}\inf_{\chi\in S_n}\|f-\chi\|_\infty. \tag{3.6}$$

A closely related result has been stated by Zygmund [29, Chapter 10, Theorem 7.1].

EXAMPLE C (The sphere in $\mathbb{R}^s$). Here

$$\Omega=\left\{x\in\mathbb{R}^s\colon \sum x_j^2=1\right\}=:\Omega_s,$$

$d\omega=$ angular measure,

$$V=\int_\Omega d\omega=|\Omega_s|,$$

$$d=\binom{s+n-1}{s-1}+\binom{s+n-2}{s-1}\ \text{(see below)}.$$

Let

$$\sum_{k=1}^{m} w_k\,g(x_k) \tag{3.7}$$

be any quadrature rule for Ω_s which satisfies (2.6) and which is exact for g a spherical polynomial of degree $\leqslant 2n$, i.e.

$$\sum_{k=1}^{m} w_k\, g(x_k) = \int_{\Omega_s} g\, d\omega \qquad \forall g \in S_{2n}. \tag{3.8}$$

(Some examples of such rules are given in Section 4).

As basis functions for the space S_n we may take spherical harmonics. The number of linearly independent spherical harmonics of exact degree l is [9, 14]

$$\delta(l) = \binom{s+l-1}{s-1} - \binom{s+l-3}{s-1}, \tag{3.9}$$

where it is understood that

$$\binom{a}{b} = 0 \qquad \text{if} \quad a < b.$$

Let $\{Y_{l1}, ..., Y_{l\delta(l)}\}$ be an orthonormal set of spherical harmonics of degree l. Then as a basis for S_n we may take

$$\{Y_{lr}: 1 \leqslant r \leqslant \delta(l),\ 0 \leqslant l \leqslant n\},$$

and the dimension of S_n is

$$d = \sum_{l=0}^{n} \delta(l) = \binom{s+n-1}{s-1} + \binom{s+n-2}{s-1}. \tag{3.10}$$

The approximation that corresponds to the quadrature rule (3.7) is

$$L_n f = \sum_{l=0}^{n} \sum_{r=1}^{\delta(l)} \sum_{k=1}^{m} w_k f(x_k)\, Y_{lr}(x_k)\, Y_{lr}. \tag{3.11}$$

By Theorem 1 this approximation satisfies

$$\left(\int_{\Omega_s} |L_n f - f|^2\, d\omega \right)^{1/2} \leqslant 2\, |\Omega_s|^{1/2} \inf_{\chi \in S_n} \|f - \chi\|_\infty. \tag{3.12}$$

Ditkin and Lyusternik [7] and Wienert [27] have previously advocated the use of particular approximations of the form (3.11). The latter paper obtained an estimate similar to (3.12) for the particular case $s = 3$.

4. The Case of the Sphere in $\mathbb{R}^s$ in More Detail

In this section we first indicate some quadrature rules over the sphere which could be used in Example C of the preceding section. The notation is that of Example C. For more general discussions of quadrature over the sphere see [25].

We shall also be interested in the number of quadrature points m required by the quadrature rule. In this connection, we note from Lemma 2 and (3.10) that

$$m \geqslant \binom{s+n-1}{s-1} + \binom{s+n-2}{s-1} = d_n, \tag{4.1}$$

a result first stated by Mysovskikh [16]. We shall see that minimal quadrature formulas (i.e. rules with $m = d_n$) do not exist for $s \geqslant 3$ and $n > 2$. It then follows from Lemma 3 that $L_n f$ can have the classical interpolation property only if $s \leqslant 2$ or $n \leqslant 2$.

4.1. *Spherical t-Designs*

A spherical t-design, a notion introduced by Delsarte, Goethals and Seidel [6] (see also [5]), is a set of points $\{x_k\}_{k=1}^m$ on the sphere Ω_s such that the equal-weight quadrature rule based on these points is exact for all polynomials of degree $\leqslant t$. Setting $t = 2n$, a spherical $2n$-design $\{x_k\}_{k=1}^m$ has the property that

$$\frac{|\Omega_s|}{m} \sum_{k=1}^{m} g(x_k) = \int_{\Omega_s} g \, d\omega \qquad \forall g \in S_{2n}, \tag{4.2}$$

so that (3.8) is satisfied with $w_k = |\Omega_s|/m$. Thus the equal weight quadrature rule based on a spherical $2n$-design is a suitable rule for use in Example C. Fortunately, it is known from the work of Seymour and Zaslavsky [20] that spherical t-designs exist for every value of t and every dimension s.

Explicit constructions of spherical t-designs are now known for arbitrary s and t, see [2, 18], but the numbers of points in these designs are thought to be far from the least possible.

That the number of points m in a spherical $2n$-design satisfies the bound (4.1) was shown already in [6]. Significantly for us, it is also known that equality cannot hold, except in special cases. A spherical $2n$-design with m points is said to be "tight" if equality holds in (4.1), that is, if

$$m = \binom{s+n-1}{s-1} + \binom{s+n-2}{s-1}. \tag{4.3}$$

Thus a spherical design is tight if and only if the corresponding equal-weight quadrature formula is minimal. Bannai and Damerell [3] show that for $s \geqslant 3$ and $n \geqslant 3$ there exist no tight spherical designs. (The non-existence of tight spherical designs for $n = 3$ had already been established in [6].) It follows then from Lemma 3 that the approximation (3.11) with $w_k = |\Omega_s|/m$ cannot be interpolatory for arbitrary functions f if $s \geqslant 3$ and $n \geqslant 3$. Thus we have established:

LEMMA 6. *For the case of a sphere in* $\mathbb{R}^s$ *with* $s \geqslant 3$, *if* $n \geqslant 3$ *and the weights* $w_1, ..., w_m$ *are all equal, then the approximation* $L_n f$ *does not have the classical interpolation property* (2.12).

The fact that spherical t-designs play an important role in Lagrange interpolation theory for spheres has been pointed out previously by Bos [4]. In particular, Theorem 6 of [4] is closely related to Lemma 6.

4.2. *Other Rules*

The special feature of spherical t-designs is that the corresponding quadrature formulas have equal weights. Once this restriction is removed, and we admit any quadrature rule satisfying (2.6) and (3.8), the possibilities are enlarged.

For example, rules of higher order may be obtained by taking an appropriately weighted average of two spherical-design rules [9]. Another possibility, explored in [12], is to restrict attention to quadrature formulas that are fully symmetric with respect to a given cartesian coordinate system, in the sense that the rule is unchanged if the cartesian coordinates of the points are permuted, or any cartesian component of the points changed in sign. Explicit formulas of degree 11 (and therefore allowing $n = 5$) are presented in [12] for $5 \leqslant s \leqslant 9$, and [11] gives formulas of degree up to 9 in any number of dimensions, and of degree up to 17 for $s = 3$ and 4.

There is a large literature, initiated by Sobolev [23], concerning rules that are invariant under a symmetry group of Ω_s. In this situation a theorem of Sobolev [23], shows that all polynomials of degree $< a$ are integrated exactly, where a is the degree of the non-trivial invariant polynomial (with respect to the particular symmetry group) of lowest degree.

Lebedev [13] has constructed a large number of quadrature rules for Ω_3 by exploiting the Sobolev theory for the particular case of octohedral symmetry, and then solving explicitly the equations (3.8) for the points and weights. In most cases the weights turn out to be positive, and in these cases the rule is a candidate for use in Example C. Lebedev's rules of order $2n$ have the property that

$$\frac{m}{d_n} = \frac{m}{(n+1)^2} \to \frac{4}{3} \quad \text{as} \quad n \to \infty, \tag{4.4}$$

so that (4.1) is not an equality, and the rules are not minimal, at least for large n.

In a different direction, Stroud [25] has proposed tensor products of Gauss rules with respect to appropriate angular variables. Since he shows that such rules integrate all spherical polynomials up to arbitrarily high degree for Gauss rules of sufficiently high order, these too are suitable quadrature rules for use in Example C. For example, for the case of Ω_3, if the integral is written in polar coordinates,

$$\int_{\Omega_3} g\, d\omega = \int_0^{2\pi} \int_0^{\pi} g(\theta, \phi) \sin\theta \, d\theta \, d\phi,$$

with θ the polar angle and ϕ the azimuthal angle, then a rule of order $2n+1 > 2n$ is

$$\frac{\pi}{n+1} \sum_{j=1}^{2(n+1)} \sum_{i=1}^{n+1} \mu_i g\left(\theta_i, \frac{j\pi}{n+1}\right),$$

where $\{\cos\theta_i\}$ are the zeros of the Legendre polynomial of degree $n+1$, and $\{\mu_i\}$ are the corresponding Gauss-Legendre weights. The value of m/d_n in this case is

$$\frac{m}{d_n} = \frac{2(n+1)^2}{(n+1)^2} = 2,$$

which, as pointed out by Atkinson [1], is 50% larger than the asymptotic value for Lebedev's construction. It is 100% larger than the ratio of a minimal quadrature rule, were such a thing to exist.

4.3. *Can Minimal Quadrature Rules for the Sphere in $\mathbb{R}^s$ Exist?*

While unequal-weight quadrature rules for the sphere might be useful, we shall see that they cannot be minimal: for we shall show that a minimal rule must have equal weights. Minimal rules are therefore subject to the very restrictive range of possibilities discussed above under the heading of spherical t-designs.

LEMMA 7. *A minimal quadrature rule for the sphere in $\mathbb{R}^s$ has equal weights.*

Proof. From the definition of a minimal rule we have $m = d_n$. As a result (2.14) gives us an explicit expression for the weights, namely

$$w_k = \left(\sum_{l=0}^{n} \sum_{r=1}^{\delta(l)} Y_{lr}(x_k)^2 \right)^{-1}, \qquad 1 \leqslant k \leqslant m = d_n.$$

Since the sum

$$\sum_{r=1}^{\delta(l)} Y_{lr}(x)^2$$

with $\delta(l)$ given by (3.9) is invariant under rotations of the sphere (see [14, p. 7]), it follows that the weights w_k are all equal, and hence, from (2.7) with $g \equiv 1$,

$$w_k = \frac{1}{d_n} |\Omega_s|. \quad \blacksquare$$

On combining the last two lemmas we obtain our final result.

THEOREM 2. *For the case of the sphere in* $\mathbb{R}^s$ *with* $s \geq 3$, *if* $n \geq 3$ *then the quadrature formula satisfying* (2.6) *and* (3.8) *is not minimal, and the approximation* $L_n f$ *given by* (3.11) *does not have the classical interpolation property* (2.12).

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